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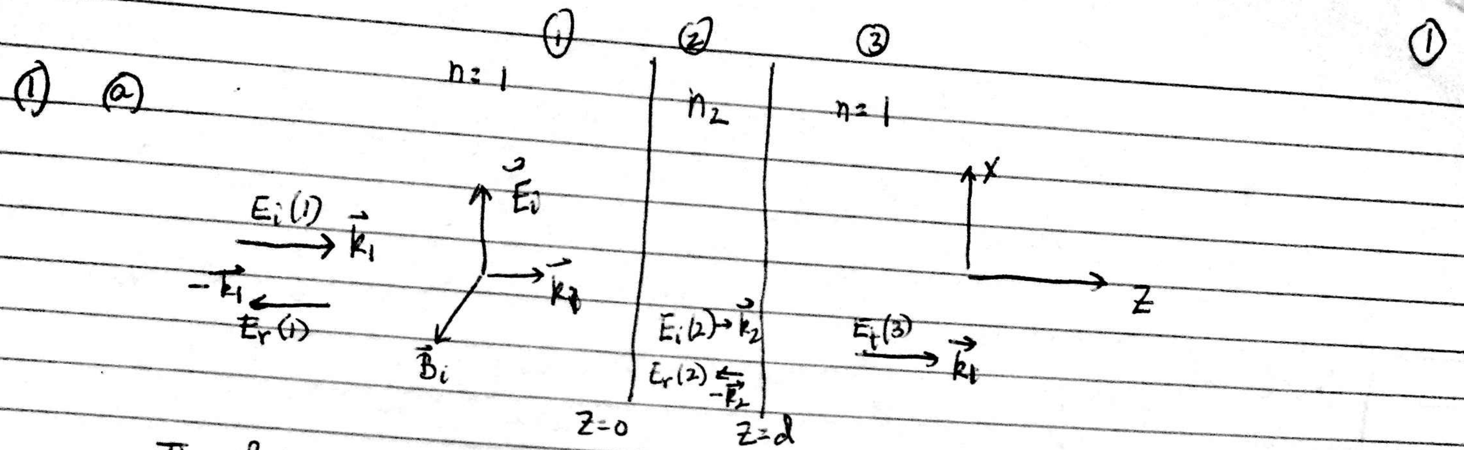
(1) (a)



The first part of this problem is identical to HW Problem (Jackson 7.2)

Let $\vec{e}_0$ be along x-axis, incident wave vector $\vec{k}_1$ along z-axis. $E_i = E_0$

Electric field is given by (leaving out common $e^{-i\omega t}$ factor).

Region (1) $\left(E_0 e^{ik_1 z} + E_r e^{-ik_1 z}\right)\vec{e}_0$

Region (2) $\left(E_i(2) e^{ik_2 z} + E_r(2) e^{-ik_2 z}\right)\vec{e}_0$

Region (3) $E_t(3)\, e^{ik_1 z}\, \vec{e}_0$ (because $k = n.\frac{\omega}{c}$, wave vector in region (3) = wave vector in region (1))

Assume all μ's are $\simeq \mu_0$ $\quad n_2 = \sqrt{\frac{\epsilon(\omega)}{\epsilon_0}}$ $\quad k_2 = n_2 k_1$ $\quad k_1 = \frac{\omega}{c}$

Boundary conditions at $z=0$

$$E_0 + E_r = E_i(2) + E_r(2) \quad \text{(from tangential } \vec{E} \text{ field)}$$

$$E_0 - E_r = n_2\left(E_i(2) - E_r(2)\right) \quad \text{(from tangential } \vec{H} \text{ field)}$$

so $$\begin{pmatrix} E_i(2) \\ E_r(2) \end{pmatrix} = \underline{\underline{t}}_1 \begin{pmatrix} E_0 \\ E_r \end{pmatrix} \; ; \quad \underline{\underline{t}}_1 \equiv \begin{pmatrix} \frac{(1+n_2)}{2n_2} & \frac{(n_2-1)}{2n_2} \\ \frac{(n_2-1)}{2n_2} & \frac{1+n_2}{2n_2} \end{pmatrix}$$

Boundary conditions at $z=d$

$$E_i(2) e^{ik_2 d} + E_r(2) e^{-ik_2 d} = E_t(3) e^{ik_1 d}$$

$$n_2\left(E_i(2) e^{ik_2 d} - E_r(2) e^{-ik_2 d}\right) = E_t(3) e^{ik_1 d}$$

or $$\begin{pmatrix} E_t(3) \\ 0 \end{pmatrix} = \underline{\underline{t}}_2 \begin{pmatrix} E_i(2) \\ E_r(2) \end{pmatrix} \qquad \underline{\underline{t}}_2 = \begin{pmatrix} \frac{n_2+1}{2} e^{i(k_2-k_1)d} & \frac{1-n_2}{2} e^{-i(k_2+k_1)d} \\ \frac{1-n_2}{2} e^{i(k_2-k_1)d} & \frac{1+n_2}{2} e^{-i(k_2+k_1)d} \end{pmatrix}$$

so $$\begin{pmatrix} E_t(3) \\ 0 \end{pmatrix} = \underset{\approx}{T}\begin{pmatrix} E_0 \\ E_r \end{pmatrix} = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}\begin{pmatrix} E_0 \\ E_r \end{pmatrix} \qquad (1)$$

$\underset{\approx}{T} = \underset{\approx}{t_2}\,\underset{\approx}{t_1}$, so using $k_2 = n_2 k_1$

$$\left.\begin{aligned}
T_{11} &= \left(\frac{n_2+1}{2}\right)\left(\frac{1+n_2}{2n_2}\right) e^{i(n_2-1)k_1 d} + \left(\frac{1-n_2}{2}\right)\left(\frac{n_2-1}{2n_2}\right) e^{-i(n_2+1)k_1 d} \\
T_{12} &= \left(\frac{n_2+1}{2}\right)\left(\frac{n_2-1}{2n_2}\right) e^{i(n_2-1)k_1 d} + \left(\frac{1-n_2}{2}\right)\left(\frac{1+n_2}{2n_2}\right) e^{-i(n_2+1)k_1 d} \\
T_{21} &= \left(\frac{1-n_2}{2}\right)\left(\frac{1+n_2}{2n_2}\right) e^{i(n_2-1)k_1 d} + \left(\frac{n_2+1}{2}\right)\left(\frac{n_2-1}{2n_2}\right) e^{i(n_2+1)k_1 d} \\
T_{22} &= \left(\frac{1-n_2}{2}\right)\left(\frac{n_2-1}{2n_2}\right) e^{i(n_2-1)k_1 d} + \left(\frac{n_2+1}{2}\right)\left(\frac{n_2+1}{2n_2}\right) e^{-i(n_2+1)k_1 d}
\end{aligned}\right\} \quad (2)$$

From Eqs (1)

$$E_r = -\frac{T_{21}}{T_{22}} = r E_0$$

$$E_t(3) = \left[T_{11} - \frac{T_{12}T_{21}}{T_{22}}\right] E_0 = t E_0$$

so transmitted wave $= \vec{e}_0 E_0 e^{ik_1 z}\left[T_{11} - \frac{T_{12}T_{21}}{T_{22}}\right]$

Now by Eqs (2), $T_{11} = \frac{(1+n_2)^2}{4n_2} e^{i(n_2-1)kd} - \frac{(1-n_2)^2}{4n_2} e^{-i(n_2+1)kd}$

(Expanding exponentials to 1st order in d) $= \frac{(1+n_2)^2}{4n_2}\left[1 + i(n_2-1)kd\right] - \frac{(1-n_2)^2}{4n_2}\left[1 - i(n_2+1)kd\right]$

after some algebra $= 1 + i(n_2^2-1)k\frac{d}{2}$.

$$T_{12} = \frac{(n_2^2-1)}{4n_2} e^{i(n_2-1)k_1 d} - \frac{(n_2^2-1)}{4n_2} e^{-i(n_2+1)kd}$$

$$\approx \frac{(n_2^2-1)}{4n_2}\left[\cancel{1} + i(n_2-1)k_1 d - \cancel{1} + i(n_2+1)kd\right]$$

$$= \frac{(n_2^2-1)}{4n_2}\, 2 i n_2 k_1 d = i\frac{(n_2^2-1)}{2} k_1 d$$

$$T_{21} = \frac{(1-n_2^2)}{4n_2} e^{i(n_2-1)k_1 d} + \frac{(n_2^2-1)}{4n_2} e^{i(n_2+1)kd}$$

$$= -T_{12} \simeq -\frac{i(n_2^2-1)}{2} k_1 d$$

$$T_{22} = -\frac{(n_2-1)^2}{4n_2} e^{i(n_2-1)k_1 d} + \frac{(n_2+1)^2}{4n_2} e^{-i(n_2+1)k_1 d}$$

$$\simeq -\frac{(n_2-1)^2}{4n_2}\left[1 + i(n_2-1)k_1 d\right] + \frac{(n_2+1)^2}{4n_2}\left[1 - i(n_2+1)kd\right]$$

$$= 1 - \frac{ikd}{2}(n_2^2+3)$$

so $\frac{T_{12}T_{21}}{T_{22}} \sim O(d^2)$

so to order d,

$$\vec{E}_t = \vec{e}_0 E_0 e^{ik_1 z}\left[1 + i(n_2^2-1)\frac{kd}{2}\right]$$

$$= \vec{e}_0 E_0 e^{ik_1 z}\left[1 + i\left(\frac{\epsilon(\omega)}{\epsilon_0} - 1\right)\frac{kd}{2}\right] \qquad (3a)$$

to order

$$\vec{E}_r = \vec{e}_0 E_0 e^{-ik_1 z}\left(-T_{21}/T_{22}\right) = \vec{e}_0 E_0 e^{-ik_1 z}\left[i(n_2^2-1)\frac{kd}{2}\right] = \vec{e}_0 E_0 e^{-ik_2}\left[i\left(\frac{\epsilon(\omega)}{\epsilon_0}-1\right)\frac{kd}{2}\right] ($$

b) If we define the magnitudes of the Poynting vectors of the incident, reflected + transmitted waves as

$$I_1 = \frac{1}{2Z_0}|\vec{E}_i|^2 \; ; \; I_2 = \frac{1}{2Z_0}|\vec{E}_r|^2 \; ; \; I_3 = \frac{1}{2Z_0}|\vec{E}_t|^2$$

(Z_0 = Impedance of free space)

then the rate of power dissipation per unit area in the slab

$$P_{abs} = I_1 - I_2 - I_3 = \frac{1}{2Z_0}\left[E_0^2 - |E_t|^2 - |E_r|^2\right] \qquad (4)$$

Because of the conductivity $\frac{\epsilon(\omega)}{\epsilon_0}$ has an imaginary part

$$\epsilon(\omega) = \epsilon_b(\omega) + \frac{i\sigma(\omega)}{\omega} \qquad \text{(Jackson Eq (7.56))} \qquad (5)$$

where $\epsilon_b(\omega)$ can be considered as real + the bound part + $\sigma(\omega)$ is the conductivity. Because of the form of Eq. (2) we must consider possible terms of $O(d^2)$ in E_t and E_r.

Thus we write

$$\vec{E}_t = \vec{e}_0 E_0 e^{ik_1 z}\left[1 + i\left(\frac{\epsilon_b}{\epsilon_0} - 1\right)\frac{kd}{2} - \frac{\sigma}{\epsilon_0\omega}\frac{kd}{2} + O(d^2)\right]$$

$$|E_t|^2 = |E_0|^2\left[1 - \frac{\sigma}{\epsilon_0\omega}kd\right]$$ to leading order in d.

From Eq. (38), $|E_r|^2 \sim O(d^2)$ so we can neglect it.

So to leading order in d,

$$P_{abs} = \frac{|E_0|^2}{2Z_0}\frac{\sigma}{\epsilon_0\omega}k\,d = \frac{1}{2Z_0}\frac{\sigma}{\epsilon_0 c}d \qquad (\omega/k = c)$$

$$= \frac{1}{2}\sqrt{\frac{\epsilon_0}{\mu_0}}\frac{\sigma}{\epsilon_0}\sqrt{\mu_0\epsilon_0}\,d\,|E_0|^2 = \boxed{\frac{1}{2}|E_0|^2\sigma d}$$

[NOTE: We could also have got this result by recognizing that time averaged ~~power dissipated~~ per unit volume in conductor $= \frac{1}{2}\vec{J}.\vec{E} = \frac{1}{2}\sigma|E|^2$!! (only true to lowest order since it assumes field in conductor is incident field)]

2.

(a) Eqns of motion for electron is

$$m\left[\ddot{\vec{x}} + \gamma\dot{\vec{x}} + \omega_0^2\vec{x}\right] - e\vec{B}_0\times\dot{\vec{x}} = -e\vec{E}e^{-i\omega t} \qquad (\vec{B}_0 \text{ along } z)$$

Write electric field $\vec{E}$ as a circularly polarized $(\pm)$ wave

$$\vec{E} = E_0(\vec{\epsilon}_1 \pm i\vec{\epsilon}_2)$$

Choose solutions of form $\vec{x} = x_0(\vec{\epsilon}_1 \pm i\vec{\epsilon}_2)e^{-i\omega t}$ $(\vec{\epsilon}_1, \vec{\epsilon}_2$ along $x, y)$

so we get $\left[-m\omega^2 - im\omega\gamma + m\omega_0^2\right]x_0(\vec{\epsilon}_1 \pm i\vec{\epsilon}_2)$

$$+ i\omega e B_0 x_0(\mp i\vec{\epsilon}_1 + \vec{\epsilon}_2) = -eE_0(\vec{\epsilon}_1 \pm i\vec{\epsilon}_2)$$

Equaling coeff. of $\vec{\epsilon}_1$,

$$m(\omega_0^2 - \omega^2 - i\gamma\omega)x_0 \pm eB_0\omega x_0 = -eE_0$$

or $$x_0 = -\frac{e}{m}\frac{E_0}{\left[\omega_0^2 - \omega^2 - i\gamma\omega \pm eB_0\omega/m\right]}$$

of $\vec{\epsilon}_2$ gives same solution

so $$\vec{P}_\pm = -e\vec{x}_\pm = \frac{e^2}{m}\frac{1}{\omega_0^2 - \omega^2 - i\gamma\omega \pm eB_0\omega/m}$$

so $$\left[\frac{\epsilon(\omega)}{\epsilon_0}\right]_\pm = 1 + \chi_e = 1 + \frac{Ne^2}{\epsilon_0 m}\frac{1}{\omega_0^2 - \omega^2 - i\gamma\omega \pm eB_0\omega/m}$$

$$\left(\frac{\epsilon(\omega)}{\epsilon_0}\right)_\pm = n_\pm^2, \text{ so } n_+^2 - n_-^2 = \frac{Ne^2}{\epsilon_0 m}\left[\frac{1}{\omega_0^2 - \omega^2 - i\gamma\omega + eB_0\omega/m} - \frac{1}{\omega_0^2 - \omega^2 - i\gamma\omega - eB_0\omega/m}\right]$$

(8) Consider a plane wave which propagates as the sum of 2 oppositely circularly polarized waves with a relative phase ~~difference~~ of ϕ.

Complex field is written as $\vec{E} = E_0\left[\frac{1}{2}(\vec{\epsilon}_1 + i\vec{\epsilon}_2) + \frac{1}{2}(\vec{\epsilon}_1 - i\vec{\epsilon}_2)e^{i\phi}\right]e^{i(kz-\omega t)}$

$\vec{\epsilon}_1, \vec{\epsilon}_2$ along $\vec{x}$ & $\vec{y}$ respectively. (1)

At $z=0$, $t=0$, actual field is the real part of (1)

If $\phi = 0$, $\vec{E} = E_0\vec{\epsilon}_1$ (plane polarized along $\hat{x}$)

If $\phi \neq 0$, $\vec{E} = \mathrm{Re}\left\{E_0\left(\frac{1+\cos\phi}{2}\right)\vec{\epsilon}_1 + \frac{1}{2}\sin\phi\,\vec{\epsilon}_2 + i\left(\frac{1}{2}\vec{\epsilon}_2 + \frac{1}{2}\vec{\epsilon}_1\sin\phi - \frac{1}{2}\vec{\epsilon}_2\cos\phi\right)\right\}$

$$= E_0\left[\vec{\epsilon}_1\cos^2\frac{\phi}{2} + \vec{\epsilon}_2\sin\frac{\phi}{2}\cos\frac{\phi}{2}\right] = E_0\cos\left(\frac{\phi}{2}\right)\left[\vec{\epsilon}_1\cos\frac{\phi}{2} + \vec{\epsilon}_2\sin\frac{\phi}{2}\right]$$

$\longrightarrow$ plane polarized wave with polarization rotated by $\boxed{\frac{\phi}{2}}$ — (counter clockwise facing wave)

$$(n_+) = \left(\frac{\epsilon_+(\omega)}{\epsilon_0}\right)^{\frac{1}{2}} \simeq 1 + \frac{1}{2}\frac{Ne^2}{\epsilon_0 m}\frac{1}{(\omega_0^2-\omega^2) + eB_0\omega/m - i\gamma\omega} \quad (2)$$

$$(n_-) = \left(\frac{\epsilon_-(\omega)}{\epsilon_0}\right)^{\frac{1}{2}} \simeq 1 + \frac{1}{2}\frac{Ne^2}{\epsilon_0 m}\frac{1}{(\omega_0^2-\omega^2) - \frac{eB_0\omega}{m} - i\gamma\omega} \quad (3)$$

Phase difference between + and − circularly polarized waves (initially zero when entering medium) given by $e^{-ikL(n_+^R - n_-^R)}$ (− relative to +)

($n_\pm^R$ = Real part of $n_\pm$)

(Imaginary part gives attenuation — neglect here)

$$\left(\frac{\phi}{2}\right) = -\frac{kL}{2}(n_+^R - n_-^R) = -\frac{kL}{2}\frac{1}{2}\frac{Ne^2}{\epsilon_0 m}\left[\frac{(\omega_0^2-\omega^2) + \frac{eB_0\omega}{m}}{\left(\omega_0^2-\omega^2+\frac{eB_0\omega}{m}\right)^2 + \gamma^2\omega^2}\right.$$

$$\left. - \frac{(\omega_0^2-\omega^2) - eB_0\omega/m}{\left(\omega_0^2-\omega^2 - \frac{eB_0\omega}{m}\right)^2 + \gamma^2\omega^2}\right] \quad \text{(Full expression)}$$

If we keep only term linear in B,

$$\left(\frac{\phi}{2}\right) \simeq -\frac{kL}{2}\frac{Ne^2}{\epsilon_0 m}\frac{eB_0\omega}{m}\frac{1}{(\omega_0^2-\omega^2)^2 + \gamma^2\omega^2}$$

so it rotates by $\left(\frac{\phi}{2}\right)$ clockwise facing wave.

3. (a) Maxwell's Eqs. in Medium assuming $e^{-i\omega t}$ time dependence can be written

$$\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} = i\omega\vec{B} \quad (1)$$

$$\nabla\times\vec{B} = \mu_0(\nabla\times\vec{H}) = \mu_0\frac{\partial\vec{D}}{\partial t} = -i\omega\mu_0\epsilon_0\left[\frac{1}{\epsilon_0}\vec{D}\right] \quad (2)$$

By (1) & (2) $\nabla\times(\nabla\times\vec{E}) = i\omega(\nabla\times\vec{B}) = \omega^2\mu_0\epsilon_0\left[\frac{1}{\epsilon_0}\vec{D}\right]$ (2)

$$\nabla(\nabla\cdot\vec{E}) - \nabla^2\vec{E} = \frac{\omega^2}{c^2}\left[\frac{1}{\epsilon_0}\vec{D}\right] \quad (3)$$

For our (partially) anisotropic medium, $\vec{D} = \epsilon_0\vec{E} + \sum_\beta \delta\epsilon_{\alpha\beta}E_\beta$ (4)

& $\frac{\omega}{c} = k$, so

Eq. (3) can be written as (writing α-component of this vector Equation explicitly)

$$\boxed{\nabla^2 E_\alpha + k^2 E_\alpha = \nabla_\alpha(\nabla\cdot\vec{E}) - k^2\sum_\beta\frac{\delta\epsilon_{\alpha\beta}}{\epsilon_0}E_\beta} \quad (5)$$

(b) In Eq (5) R.H.S. can be regarded as a weak perturbation so formal solution in terms of Green's Function (Eq (10.24) of Jackson) can be written as

$$E_\alpha = E_\alpha^{(0)} + \frac{1}{4\pi}\int d^3x'\,\frac{e^{ik|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|}\left\{\nabla_\alpha(\nabla\cdot\vec{E}) - k^2\sum_\beta\frac{\delta\epsilon_{\alpha\beta}}{\epsilon_0}E_\beta\right\} \quad (6)$$

Far field solution is of form

$$E_\alpha \longrightarrow E_\alpha^{(0)} + \vec{A}_{sc}\frac{e^{ikr}}{r}$$ (by analogy with (10.26) of Jackson)

where

$$\vec{A}_{sc} = \frac{1}{4\pi}\int d^3x'\,e^{-ik\vec{n}\cdot\vec{x}'}\left\{\nabla_\alpha(\nabla\cdot\vec{E}) - k^2\sum_{\alpha\beta}\frac{\delta\epsilon_{\alpha\beta}}{\epsilon_0}E_\beta\right\} \quad (7)$$

First term can be reduced to a surface integral over finite size of medium & vanishes since $\nabla\cdot\vec{E} = 0$ in free space

In Born Approx^n choose E_β to be incident field, $\frac{E_\beta^{(0)}}{E_0} = (e_0)_\beta e^{ik\vec{n}_0\cdot\vec{x}'}$

Eq^n (7) gives

$$\boxed{\frac{\vec{A}_{sc}\cdot\vec{e}_{sc}}{E_0} = -\frac{k^2}{4\pi}\int d^3x'\,e^{i\vec{q}\cdot\vec{x}'}\sum_{\alpha\beta}(e_{sc})_\alpha\frac{\delta\epsilon_{\alpha\beta}}{\epsilon_0}(e_0)_\beta}$$

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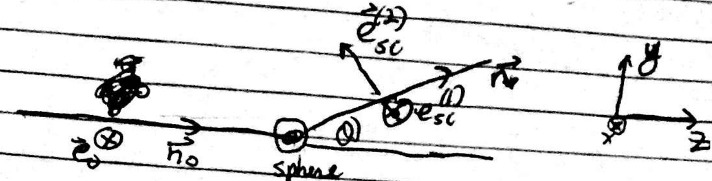
(b)



In this case, $\vec{e}_0 = (1,0,0)$; $e^{(1)}_{sc} = (1,0,0)$; $e^{(2)}_{sc} = (0, \cos\theta, -\sin\theta)$

$$\vec{q} = k(\vec{n} - \vec{n}_0) \qquad |\vec{q}| = q = 2k\sin\left(\frac{\theta}{2}\right)$$

$$\left(\frac{d\sigma}{d\Omega}\right)^{(1)} = \frac{|\vec{A}_{sc} \cdot e^{(1)}_{sc}|}{|E^0|^2} = \frac{k^4}{16\pi^2}\left|\frac{\delta\epsilon_{xx}}{\epsilon_0}\right|^2 \left|\int_{\text{sphere}} e^{i\vec{q}\cdot\vec{x}'} d^3x'\right|^2$$

from given form for $\frac{\delta\epsilon_{\alpha\beta}}{\epsilon_0}$ $\rightarrow$
$$= \frac{k^4}{16\pi^2}\frac{A^2}{\epsilon_0^2}\left[\frac{4\pi a^3}{3}\,\frac{\sin(qa) - (qa)\cos(qa)}{q^3a^3}\right]^2$$

$$= \frac{k^4 A^2}{9\epsilon_0^2}a^3\left[\frac{\sin(qa) - (qa)\cos(qa)}{q^3a^3}\right]^2 \quad \text{for } \vec{e}_{sc} \parallel \text{ to } x\text{-axis}$$

$$\left(\frac{d\sigma}{d\Omega}\right)^{(2)} = \frac{k^4}{16\pi^2}\left|\frac{\delta\epsilon_{yx}}{\epsilon_0} + \frac{\delta\epsilon_{zx}}{\epsilon_0}\right|^2 \left|\int_{\text{sphere}} e^{i\vec{q}\cdot\vec{x}'} d^3x'\right|^2$$

$$= \frac{k^4}{16\pi^2}\left[i B \cancel{M_x} - iB\cancel{M_y}\right|^2 \left|\int_{\text{sphere}} e^{i\vec{q}\cdot\vec{x}'} d^3x'\right|^2$$

(only M_z exists) $= 0$ for $\vec{e}_{sc} \perp$ to x-axis

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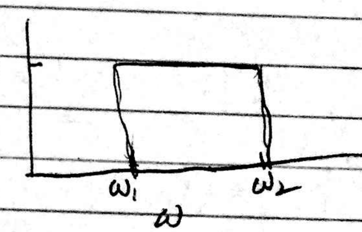
(4) $$\text{Im}\left(\frac{\epsilon(\omega)}{\epsilon_0}\right) = \lambda\left(\theta(\omega-\omega_1) - \theta(\omega-\omega_2)\right)$$



By Kramers-Kronig. Eq. (7.120) of Jackson

$$\text{Re}\left(\frac{\epsilon(\omega)}{\epsilon_0}\right) = 1 + \frac{2}{\pi}P\int_0^\infty \frac{\omega'\,\text{Im}(\epsilon(\omega')/\epsilon_0)}{\omega'^2 - \omega^2}d\omega'$$

$$= 1 + \frac{2\lambda}{\pi}P\int_{\omega_1}^{\omega_2}\frac{\omega'}{\omega'^2 - \omega^2}d\omega' = 1 + \frac{\lambda}{\pi}\left[\ln\left|\frac{\omega_2^2 - \omega^2}{\omega_1^2 - \omega^2}\right|\right]$$

(substitute $z = \omega'^2$)